REPRESENTATION OF UP-ALGEBRAS IN INTERVAL-VALUED INTUITIONISTIC FUZZY ENVIRONMENT

Tapan Senapati*
Department of Applied Mathematics with Oceanology and Computer Programming
Vidyasagar University
Midnapore 721102
India
math.tapan@gmail.com

G. Muhiuddin
Department of Mathematics
University of Tabuk
Tabuk 71491
Saudi Arabia
chishtygm@gmail.com

K.P. Shum
Institute of Mathematics
Yunnan University
Kunming 650091
People's Republic of China
emailkpshum@ynu.edu.cn

Abstract. In this paper, the concept of interval-valued intuitionistic fuzzy set to UP-subalgebras and UP-ideals of UP-algebras are introduced. Relations among IVIF UP-subalgebras with IVIF UP-ideals of UP-algebras are investigated. The homomorphic image and inverse image of IVIF UP-subalgebras and IVIF UP-ideals are studied and some related properties are investigated. Equivalence relations on IVIF UP-ideals are discussed. Also, the product of IVIF UP-algebras are investigated.

Keywords: UP-algebra, interval-valued intuitionistic fuzzy set, interval-valued intuitionistic fuzzy UP-subalgebra, interval-valued intuitionistic fuzzy UP-ideal, equivalence relation, upper(lower)-level cuts, product of UP-algebra.

1. Introduction

The theory of the fuzzy set introduced by Zadeh has achieved a great success in various fields. Atanassov [1] introduced the intuitionistic fuzzy set (IFS), which is a generalization of the fuzzy set. The IFS has received more and more attention and has been applied to many fields since its appearance. The theory of the IFS has been found to be more useful to deal with vagueness and uncertainty

∗. Corresponding author

in decision situations than that of the fuzzy set. Atanassov and Gargov further generalized the IFS in the spirit of ordinary interval-valued fuzzy sets (IVFSs) and defined the notion of an interval-valued intuitionistic fuzzy set (IVIFS).

BCK-algebras and BCI-algebras [4] are two important classes of logical algebras introduced by Imai and Iseki. Neggers and Kim [9] introduced a new notion, called a B-algebras which is related to several classes of algebras of interest such as BCK/BCI-algebras. Kim and Kim [8] introduced the notion of BG-algebras, which is a generalization of B-algebras. Senapati together with colleagues [2, 5, 6, 10-23] have done lot of works on BCK/BCI-algebras and related algebras. Iampan [3] introduced a new branch of logical algebra called UP-algebras, which is related to $BCK/BCI/B/BG$-algebras. Somjanta et al. [24] applied the concept of fuzzy set theory to UP-algebra. Kesorn et al. [7] introduced intuitionistic fuzzy UP-algebras and discussed their properties in details.

The objective of this paper is to introduce the concept of Atanassov's interval-valued intuitionistic fuzzy sets in UP-algebras. The images and preimages of IVIF UP-subalgebras and UP-ideals has been introduced and some important properties of it are also studied. The rest of the paper is organized as follows. Section 2 recalls some definitions, viz., UP-algebra, UP-subalgebra, UP-ideal and refinement of unit interval. In Section 3, UP-subalgebras of IVIFSs are defined with some its properties. In next Section, IVIF UP-ideals are defined and related properties are investigated. In Section 5, homomorphism of IVIF UP-subalgebras and UP-ideals, and some of its properties are studied. In Section 6, equivalence relations on IVIF UP-ideals are introduced. In Section 7, product of IVIF UP-subalgebras and UP-ideals are investigated. Finally, in Section 8, a conclusion of the proposed work is given.

2. Preliminaries

Here we give a brief review of some preliminaries.

Definition 2.1 ([3]). *By a UP-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ with a single binary operation $*$ that satisfies the following axioms: for any $x, y, z \in X$,*

1. $(y * z) * ((x * y) * (x * z)) = 0$,
2. $0 * x = x$,
3. $x * 0 = 0$,
4. $x * y = y * x = 0$ *implies* $x = y$.

In what follows, let $(X, *, 0)$ denote a UP-algebra unless otherwise specified. For brevity we also call X a UP-algebra. We can define a partial ordering "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2 ([7]). *In a UP-algebra, the following axioms are true: for any $x, y, z \in X$,*

(i) $x * x = 0$,

(ii) $x * y = y * z = 0$ *implies* $x * z = 0$,

(iii) $x * y = 0$ *implies* $(z * x) * (z * y) = 0$,

(iv) $x * y = 0$ *implies* $(y * z) * (x * z) = 0$,

(v) $x * (y * x) = 0$,

(vi) $(y * x) * x = 0$ *if and only if* $x = y$,

(vi) $x * (y * y) = 0$.

A non-empty subset S of a UP-algebra X is called a UP-subalgebra [7] of X if $x * y \in S$, for all $x, y \in S$. From this definition it is observed that, if a subset S of a UP-algebra satisfies only the closer property, then S becomes a UP-subalgebra.

A nonempty subset T of X is called an UP-ideal [3] of X if it satisfies the following properties: (I_1) the constant $0 \in T$, (I_2) for ant $x, y, z \in X$, $x * (y * z) \in T$ and $y \in T \Rightarrow x * z \in T$.

Let $(X, *, 0)$ and $(Y, *', 0')$ be UP-algebras. A homomorphism is a mapping $f : X \to Y$ satisfying $f(x * y) = f(x) *' f(y)$, for all $x, y \in X$.

Let $D[0, 1]$ be the set of all closed subintervals of the interval $[0, 1]$. Consider two elements $D_1, D_2 \in D[0, 1]$. If $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$, then $rmin(D_1, D_2) = [\min(a_1, a_2), \min(b_1, b_2)]$ which is denoted by $D_1 \wedge^r D_2$ and $rmax(D_1, D_2) = [\max(a_1, a_2), \max(b_1, b_2)]$ which is denoted by $D_1 \vee^r D_2$. Thus, if $D_i = [a_i, b_i] \in D[0, 1]$ for $i = 1, 2, 3, 4, \ldots$, then we define $rsup_i(D_i) = [\sup_i(a_i), \sup_i(b_i)]$, i.e, $\vee_i^r D_i = [\vee_i a_i, \vee_i b_i]$. Similarly, we define $rinf_i(D_i) = [\inf_i(a_i), \inf_i(b_i)]$ i.e, $\wedge_i^r D_i = [\wedge_i a_i, \wedge_i b_i]$. Now we call $D_1 \geq D_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly, the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

Our main objective is to investigate the idea of UP-subalgebras and UP-ideals of IVIFS. The IVIFS is a particular type of fuzzy set.

Definition 2.3 ([25])**.** (*Fuzzy Set*) *Let X be the collection of objects denoted generally by x then a fuzzy set A in X is defined as $A = \{< x, \mu_A(x) >: x \in X\}$ where $\mu_A(x)$ is called the membership value of x in A and $0 \leq \mu_A(x) \leq 1$.*

Combined the definition of UP-subalgebra and UP-ideal over crisp set and the idea of fuzzy set Somjanta et al. [24] defined fuzzy UP-subalgebra and UP-ideal, which is defined below.

Definition 2.4 ([24])**.** *Let $A = \{< x, \mu_A(x) >: x \in X\}$ be a fuzzy set in a UP-algebra. Then A is called a fuzzy UP-subalgebra of X if $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.*

A is called a fuzzy UP-ideal of X if $\mu_A(0) \geq \mu_A(x)$ and $\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$ for all $x, y, z \in X$.

Definition 2.5 ([1]). (*IVIFS*) *An IVIFS A over X is an object having the form $A = \{\langle x, R_A(x), Q_A(x)\rangle : x \in X\}$, where $R_A(x) : X \to D[0,1]$ and $Q_A(x) : X \to D[0,1]$. The intervals $R_A(x)$ and $Q_A(x)$ denote the intervals of the degree of belongingness and non-belongingness of the element x to the set A, where $R_A(x) = [R_{AL}(x), R_{AU}(x)]$ and $Q_A(x) = [Q_{AL}(x), Q_{AU}(x)]$, for all $x \in X$, with the condition $0 \leq R_{AU}(x) + Q_{AU}(x) \leq 1$. For the sake of simplicity, we shall use the symbol $A = (R_A, Q_A)$ for the IVIFS $A = \{\langle x, R_A(x), Q_A(x)\rangle : x \in X\}$.*

Also note that $\overline{R}_A(x) = [1-R_{AU}(x), 1-R_{AL}(x)]$ and $\overline{Q}_A(x) = [1-Q_{AU}(x), 1-Q_{AL}(x)]$, where $[\overline{R}_A(x), \overline{Q}_A(x)]$ represents the complement of x in A.

3. IVIF UP-subalgebras of UP-algebras

In this section, we will introduce a new notion called interval-valued intuitionistic fuzzy UP-subalgebra (IVIF UP-subalgebra) of UP-algebras and study several properties of it.

Definition 3.1. *Let $A = (R_A, Q_A)$ be an IVIFS in X, where X is a UP-subalgebra, then the set A is IVIF UP-subalgebra over the binary operator $*$ if it satisfies the following conditions:*

$$(UP1) \qquad R_A(x * y) \geq rmin\{R_A(x), R_A(y)\},$$
$$(UP2) \qquad Q_A(x * y) \leq rmax\{Q_A(x), Q_A(y)\},$$

for all $x, y \in X$.

We consider an example of IVIF UP-subalgebra below.

Example 3.2. Let X=$\{0, a, b, c\}$ be a UP-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Define an IVIFS $A = (R_A, Q_A)$ in X by

$$R_A(x) = \begin{cases} [0.5, 0.6], & \text{if } x \in \{0, a, b\} \\ [0.1, 0.2], & \text{if } x = c \end{cases} \quad \text{and} \quad \mathrm{Q_A(x)} = \begin{cases} [0.3, 0.4], & \text{if } x \in \{0, a, b\} \\ [0.4, 0.5], & \text{if } x = c. \end{cases}$$

By routine calculations we get A is an IVIF UP-subalgebra of X.

Proposition 3.3. *If $A = (R_A, Q_A)$ is an IVIF UP-subalgebra in X, then for all $x \in X$, $R_A(0) \geq R_A(x)$ and $Q_A(0) \leq Q_A(x)$.*

Proof. It is easy and omitted. □

Theorem 3.4. *Let A be an IVIF UP-subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that* $\lim_{n\to\infty} R_A(x_n) = [1,1]$ *and* $\lim_{n\to\infty} Q_A(x_n) = [0,0]$*. Then $R_A(0) = [1,1]$ and $Q_A(0) = [0,0]$.*

Proof. By Proposition 3.3, $R_A(0) \geq R_A(x)$ for all $x \in X$, therefore, $R_A(0) \geq R_A(x_n)$ for every positive integer n. Consider, $[1,1] \geq R_A(0) \geq \lim_{n\to\infty} R_A(x_n) = [1,1]$. Hence, $R_A(0) = [1,1]$.

Again, by Proposition 3.3, $Q_A(0) \leq Q_A(x)$ for all $x \in X$, thus $Q_A(0) \leq Q_A(x_n)$ for every positive integer n. Now, $[0,0] \leq Q_A(0) \leq \lim_{n\to\infty} Q_A(x_n) = [0,0]$. Hence, $Q_A(0) = [0,0]$. $\square$

Proposition 3.5. *If an IVIFS $A = (R_A, Q_A)$ in X is an IVIF UP-subalgebra, then for all $x \in X$, $R_A(0 * x) \geq R_A(x)$ and $Q_A(0 * x) \leq Q_A(x)$.*

Proof. For all $x \in X$, $R_A(0 * x) \geq rmin\{R_A(0), R_A(x)\} = rmin\{R_A(x * x), R_A(x)\} \geq rmin\{rmin\{R_A(x), R_A(x)\}, R_A(x)\} = R_A(x)$ and $Q_A(0 * x) \leq rmax\{Q_A(0), Q_A(x)\} = rmax\{Q_A(x * x), Q_A(x)\} \leq rmax\{rmax\{Q_A(x), Q_A(x)\}, Q_A(x)\} = Q_A(x)$. This completes the proof. $\square$

Theorem 3.6. *An IVIFSs $A = \{[R_{AL}, R_{AU}], [Q_{AL}, Q_{AU}]\}$ in X is an IVIF UP-subalgebra of X if and only if R_{AL}, R_{AU}, Q_{AL} and Q_{AU} are fuzzy UP-subalgebras of X.*

Proof. Let R_{AL} and R_{AU} be fuzzy UP-subalgebra of X and $x, y \in X$. Then $R_{AL}(x * y) \geq \min\{R_{AL}(x), R_{AL}(y)\}$ and $R_{AU}(x * y) \geq \min\{R_{AU}(x), R_{AU}(y)\}$. Now,

$$\begin{aligned} R_A(x * y) &= [R_{AL}(x * y), R_{AU}(x * y)] \\ &\geq [min\{R_{AL}(x), R_{AL}(y)\}, min\{R_{AU}(x), R_{AU}(y)\}] \\ &= rmin\Big\{[R_{AL}, (x), R_{AU}(x)], [R_{AL}(y), R_{AU}(y)]\Big\} \\ &= rmin\{R_A(x), R_A(y)\}. \end{aligned}$$

Again, let Q_{AL} and Q_{AU} be fuzzy UP-subalgebras of X and $x, y \in X$. Then $Q_{AL}(x * y) \leq \max\{Q_{AL}(x), Q_{AL}(y)\}$ and $Q_{AU}(x * y) \leq \max\{Q_{AU}(x), Q_{AU}(y)\}$. Now,

$$\begin{aligned} Q_A(x * y) &= [Q_{AL}(x * y), Q_{AU}(x * y)] \\ &\leq [\max\{Q_{AL}(x), Q_{AL}(y)\}, \max\{Q_{AU}(x), Q_{AU}(y)\}] \\ &= rmax\Big\{[Q_{AL}(x), Q_{AU}(x)], [Q_{AL}(y), Q_{AU}(y)]\Big\} \\ &= rmax\{Q_A(x), Q_A(y)\}. \end{aligned}$$

Hence, $A = \{[R_{AL}, R_{AU}], [Q_{AL}, Q_{AU}]\}$ is an IVIF UP-subalgebra of X.

Conversely, assume that, A is an IVIF UP-subalgebra of X. For any $x, y \in X$

$$\begin{aligned}
[R_{AL}(x*y), R_{AU}(x*y)] &= R_A(x*y) \geq rmin\{R_A(x), R_A(y)\} \\
&= rmin\{[R_{AL}(x), R_{AU}(x)], [R_{AL}(y), R_{AU}(y)] \\
&= [\min\{R_{AL}(x), R_{AL}(y)\}, \min\{R_{AU}(x), R_{AU}(y)\}], \\
[Q_{AL}(x*y), Q_{AU}(x*y)] &= Q_A(x*y) \leq rmax\{Q_A(x), Q_A(y)\} \\
&= rmax\{[Q_{AL}(x), Q_{AU}(x)], [Q_{AL}(y), Q_{AU}(y)]\} \\
&= [\max\{Q_{AL}(x), Q_{AL}(y)\}, \max\{Q_{AU}(x), Q_{AU}(y)\}].
\end{aligned}$$

Thus $R_{AL}(x*y) \geq \min\{R_{AL}(x), R_{AL}(y)\}$, $R_{AU}(x*y) \geq \min\{R_{AU}(x), R_{AU}(y)\}$, $Q_{AL}(x*y) \leq \max\{Q_{AL}(x), Q_{AL}(y)\}$ and $Q_{AU}(x*y) \leq \max\{Q_{AU}(x), Q_{AU}(y)\}$. Therefore, R_{AL}, R_{AU}, Q_{AL} and Q_{AU} are fuzzy UP-subalgebras of X. □

Definition 3.7. *Let A and B be two IVIFSs on X, where $A = \{\langle [R_{AL}(x), R_{AU}(x)], [Q_{AL}(x), Q_{AU}]\rangle : x \in X\}$ and $B = \{\langle [R_{BL}(x), R_{BU}(x)], [Q_{BL}(x), Q_{BU}]\rangle : x \in X\}$. Then the intersection of A and B is denoted by $A \cap B$, and is given by $A \cap B = \{\langle x, R_{A\cap B}(x), Q_{A\cup B}(x)\rangle : x \in X\} = \{\langle [min(R_{AL}(x), R_{BL}(x)), min(R_{AU}(x), R_{BU}(x))], [max(Q_{AL}(x), Q_{BL}(x)), max(Q_{AU}(x), Q_{BU}(x))]\rangle : x \in X\}$.*

Theorem 3.8. *Let A_1 and A_2 be two IVIF UP-subalgebras of X. Then $A_1 \cap A_2$ is an IVIF UP-subalgebra of X.*

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2. Since A_1 and A_2 are IVIF UP-subalgebras of X, by Theorem 3.6,

$$\begin{aligned}
R_{A_1\cap A_2}(x*y) &= [R_{(A_1\cap A_2)L}(x*y), R_{(A_1\cap A_2)U}(x*y)] \\
&= [\min(R_{A_1L}(x*y), R_{A_2L}(x*y)), \\
&\qquad \min(R_{A_1U}(x*y), R_{A_2U}(x*y))] \\
&\geq [\min(R_{(A_1\cap A_2)L}(x), R_{(A_1\cap A_2)L}(y)), \\
&\qquad \min(R_{(A_1\cap A_2)U}(x), R_{(A_1\cap A_2)U}(y))] \\
&= rmin\{R_{A_1\cap A_2}(x), R_{A_1\cap A_2}(y)\} \\
\text{and } Q_{A_1\cup A_2}(x*y) &= [Q_{(A_1\cup A_2)L}(x*y), Q_{(A_1\cup A_2)U}(x*y)] \\
&= [\max(Q_{A_1L}(x*y), Q_{A_2L}(x*y)), \\
&\qquad \max(Q_{A_1U}(x*y), Q_{A_2U}(x*y))] \\
&\leq [\max(Q_{(A_1\cup A_2)L}(x), Q_{(A_1\cup A_2)L}(y)), \\
&\qquad \max(Q_{(A_1\cup A_2)U}(x), Q_{(A_1\cup A_2)U}(y))] \\
&= rmax\{Q_{A_1\cup A_2}(x), Q_{A_1\cup A_2}(y)\}.
\end{aligned}$$

This proves the theorem. □

Corollary 3.9. *Let $\{A_i | i = 1, 2, 3, 4, \ldots\}$ be a family of IVIF UP-subalgebra of X. Then $\bigcap A_i$ is also an IVIF UP-subalgebra of X where, $\bigcap A_i = \{\langle x, rminR_{A_i}(x), rmaxQ_{A_i}(x)\rangle : x \in X\}$.*

Theorem 3.10. *Let $A = (R_A, Q_A)$ be an IVIF UP-subalgebra of X and let $n \in \mathbb{N}$ (the set of natural numbers). Then*

(*i*) $R_A(\prod^n x * x) \geq R_A(x)$, *for any odd number n,*

(*ii*) $Q_A(\prod^n x * x) \leq Q_A(x)$, *for any odd number n,*

(*iii*) $R_A(\prod^n x * x) = R_A(x)$, *for any even number n,*

(*iv*) $Q_A(\prod^n x * x) = Q_A(x)$, *for any even number n.*

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2p-1$ for some positive integer p. We prove the Theorem by induction.

Now $R_A(x*x) = R_A(0) \geq R_A(x)$ and $Q_A(x*x) = Q_A(0) \leq Q_A(x)$. Suppose that $R_A(\prod^{2p-1} x*x) \geq R_A(x)$ and $Q_A(\prod^{2p-1} x*x) \leq Q_A(x)$. Then by assumption, $R_A(\prod^{2(p+1)-1} x * x) = R_A(\prod^{2p+1} x * x) = R_A(\prod^{2p-1} x * (x * (x * x))) = R_A(\prod^{2p-1} x * x) \geq R_A(x)$ and $Q_A(\prod^{2(p+1)-1} x * x) = Q_A(\prod^{2p+1} x * x) = Q_A(\prod^{2p-1} x * (x * (x * x))) = Q_A(\prod^{2p-1} x * x) \leq Q_A(x)$, which proves (i) and (ii). Proofs are similar for the cases (iii) and (iv). □

We define two operators $\bigoplus A$ and $\bigotimes A$ on IVIFS as follows:

Definition 3.11. *Let $A = (R_A, Q_A)$ be an IVIFS defined on X. The operators $\bigoplus A$ and $\bigotimes A$ are defined as $\bigoplus A = \{\langle x, R_A(x), \overline{R}_A(x)\rangle : x \in X\}$ and $\bigotimes A = \{\langle x, \overline{Q}_A(x), Q_A(x)\rangle : x \in X\}$.*

Theorem 3.12. *If $A = (R_A, Q_A)$ is an IVIF UP-subalgebra of X, then*

(*i*) $\bigoplus A$, *and*

(*ii*) $\bigotimes A$, *both are IVIF UP-subalgebras.*

Proof. For (i), it is sufficient to show that $\overline{R}_A$ satisfies the condition ($UP2$). Let $x, y \in X$. Then $\overline{R}_A(x * y) = [1,1] - R_A(x * y) \leq [1,1] - rmin\{R_A(x), R_A(y)\} = rmax\{1 - R_A(x), 1 - R_A(y)\} = rmax\{\overline{R}_A(x), \overline{R}_A(y)\}$. Hence, $\bigoplus A$ is an IVIF UP-subalgebra of X.

For (ii), it is sufficient to show that $\overline{Q}_A$ satisfies the condition ($UP1$). Let $x, y \in X$. Then $\overline{Q}_A(x * y) = [1,1] - Q_A(x * y) \geq [1,1] - rmax\{Q_A(x), Q_A(y)\} = rmin\{1 - Q_A(x), 1 - Q_A(y)\} = rmin\{\overline{Q}_A(x), \overline{Q}_A(y)\}$. Hence, $\bigotimes A$ is also an IVIF UP-subalgebra of X. □

The sets $\{x \in X : R_A(x) = R_A(0)\}$ and $\{x \in X : Q_A(x) = Q_A(0)\}$ are denoted by I_{R_A} and I_{Q_A} respectively. These two sets are also UP-subalgebra of X.

Theorem 3.13. *Let $A = (R_A, Q_A)$ be an IVIF UP-subalgebra of X, then the sets I_{R_A} and I_{Q_A} are UP-subalgebras of X.*

Proof. Let $x, y \in I_{R_A}$. Then $R_A(x) = R_A(0) = R_A(y)$ and so, $R_A(x * y) \geq rmin\{R_A(x), R_A\ (y)\} = R_A(0)$. By using Proposition 3.3, we know that $R_A(x * y) = R_A(0)$ or equivalently $x * y \in I_{R_A}$.

Again, let $x, y \in I_{Q_A}$. Then $Q_A(x) = Q_A(0) = Q_A(y)$ and so, $Q_A(x * y) \leq rmax\{Q_A(x), Q_A\ (y)\} = Q_A(0)$. Again, by Proposition 3.3, we know that $Q_A(x * y) = Q_A(0)$ or equivalently $x * y \in I_{Q_A}$.

Hence, the sets I_{R_A} and I_{Q_A} are UP-subalgebras of X. □

Theorem 3.14. *Let B be a nonempty subset of X and $A = (R_A, Q_A)$ be an IVIFS in X defined by*

$$R_A(x) = \begin{cases} [\alpha_1, \alpha_2], & \text{if } x \in B \\ [\beta_1, \beta_2], & \text{otherwise} \end{cases} \quad \textit{and} \quad Q_A(x) = \begin{cases} [\gamma_1, \gamma_2], & \text{if } x \in B \\ [\delta_1, \delta_2], & \text{otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]$ and $[\delta_1, \delta_2] \in D[0, 1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$ and $[\gamma_1, \gamma_2] \leq [\delta_1, \delta_2]$ and $\alpha_2 + \gamma_2 \leq 1$; $\beta_2 + \delta_2 \leq 1$. Then A is an IVIF UP-subalgebra of X if and only if B is a UP-subalgebra of X. Moreover, $I_{R_A} = B = I_{Q_A}$.

Proof. Let A be an IVIF UP-subalgebra of X and $x, y \in X$ be such that $x, y \in B$. Then $R_A(x * y) \geq rmin\{R_A(x), R_A(y)\} = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$ and $Q_A(x * y) \leq rmax\{Q_A(x), Q_A(y)\} = rmax\{[\gamma_1, \gamma_2], [\gamma_1, \gamma_2]\} = [\gamma_1, \gamma_2]$. So $x * y \in B$. Hence, B is a UP-subalgebra of X.

Conversely, suppose that B is a UP-subalgebra of X. Let $x, y \in X$. Consider two cases:

Case (i). If $x, y \in B$ then $x * y \in B$, thus $R_A(x * y) = [\alpha_1, \alpha_2] = rmin\{R_A(x),\ R_A(y)\}$ and $Q_A(x * y) = [\gamma_1, \gamma_2] = rmax\{Q_A(x), Q_A(y)\}$.

Case (ii). If $x \notin B$ or, $y \notin B$, then $R_A(x*y) \geq [\beta_1, \beta_2] = rmin\{R_A(x), R_A(y)\}$ and $Q_A(x * y) \leq [\delta_1, \delta_2] = rmax\{Q_A(x), Q_A(y)\}$.

Hence, A is an IVIF UP-subalgebra of X.

Now, $I_{R_A} = \{x \in X, R_A(x) = R_A(0)\} = \{x \in X, R_A(x) = [\alpha_1, \alpha_2]\} = B$ and $I_{Q_A} = \{x \in X, Q_A(x) = Q_A(0)\} = \{x \in X, Q_A(x) = [\gamma_1, \gamma_2]\} = B$. □

Definition 3.15. *Let $A = (R_A, Q_A)$ is an IVIF UP-subalgebra of X. For $[s_1, s_2], [t_1, t_2] \in D[0, 1]$, the set $U(R_A : [s_1, s_2]) = \{x \in X : R_A(x) \geq [s_1, s_2]\}$ is called upper $[s_1, s_2]$-level of A and $L(Q_A : [t_1, t_2]) = \{x \in X : Q_A(x) \leq [t_1, t_2]\}$ is called lower $[t_1, t_2]$-level of A.*

Theorem 3.16. *If $A = (R_A, Q_A)$ is an IVIF UP-subalgebra of X, then the upper $[s_1, s_2]$-level and lower $[t_1, t_2]$-level of A are subalgebras of X.*

Proof. Let $x, y \in U(R_A : [s_1, s_2])$. Then $R_A(x) \geq [s_1, s_2]$ and $R_A(y) \geq [s_1, s_2]$. It follows that $R_A(x*y) \geq rmin\{R_A(x), R_A(y)\} \geq [s_1, s_2]$ so that $x*y \in U(R_A : [s_1, s_2])$. Hence, $U(R_A : [s_1, s_2])$ is a subalgebra of X.

Let $x, y \in L(Q_A : [t_1, t_2])$. Then $Q_A(x) \leq [t_1, t_2]$ and $Q_A(y) \leq [t_1, t_2]$. It follows that $Q_A(x * y) \leq rmax\{Q_A(x), Q_A(y)\} \leq [t_1, t_2]$ so that $x * y \in L(Q_A : [t_1, t_2])$. Hence, $L(Q_A : [t_1, t_2])$ is a subalgebra of X. □

Theorem 3.17. *Let $A = (R_A, Q_A)$ be an IVIFS in X, such that the sets $U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ are subalgebras of X for every $[s_1, s_2], [t_1, t_2] \in D[0, 1]$. Then A is an IVIF UP-subalgebra of X.*

Proof. Let for every $[s_1, s_2], [t_1, t_2] \in D[0, 1], U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ are subalgebras of X. In contrary, let $x_0, y_0 \in X$ be such that $R_A(x_0 * y_0) < rmin\{R_A(x_0), R_A(y_0)\}$. Let $R_A(x_0) = [\vartheta_1, \vartheta_2]$, $R_A(y_0) = [\vartheta_3, \vartheta_4]$ and $R_A(x_0 * y_0)=[s_1, s_2]$. Then $[s_1, s_2]<rmin\{[\vartheta_1, \vartheta_2], [\vartheta_3, \vartheta_4]\}=[\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}]$. So, $s_1 < \min\{\vartheta_1, \vartheta_3\}$ and $s_2 < \min\{\vartheta_2, \vartheta_4\}$. Consider,

$$\begin{aligned}
[\rho_1, \rho_2] &= \frac{1}{2}\Big[R_A(x_0 * y_0) + rmin\{R_A(x_0), R_A(y_0)\}\Big] \\
&= \frac{1}{2}\Big[[s_1, s_2] + [\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}]\Big] \\
&= \Big[\frac{1}{2}(s_1 + \min\{\vartheta_1, \vartheta_3\}), \frac{1}{2}(s_2 + \min\{\vartheta_2, \vartheta_4\})\Big].
\end{aligned}$$

Therefore, $\min\{\vartheta_1, \vartheta_3\} > \rho_1 = \frac{1}{2}(s_1+\min\{\vartheta_1, \vartheta_3\}) > s_1$ and $\min\{\vartheta_2, \vartheta_4\} > \rho_2 = \frac{1}{2}(s_2+\min\{\vartheta_2, \vartheta_4\}) > s_2$. Hence, $[\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}] > [\rho_1, \rho_2] > [s_1, s_2]$, so that $x_0 * y_0 \notin U(R_A : [s_1, s_2])$ which is a contradiction, since $R_A(x_0) = [\vartheta_1, \vartheta_2]\geq[\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}]>[\rho_1, \rho_2]$ and $R_A(y_0)=[\vartheta_3, \vartheta_4]\geq[\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}] > [\rho_1, \rho_2]$. This implies $x_0*y_0 \in U(R_A : [s_1, s_2])$. Thus $R_A(x*y) \geq rmin\{R_A(x), R_A(y)\}$ for all $x, y \in X$.

Again, in contrary, let $x_0, y_0 \in X$ be such that $Q_A(x_0*y_0) > rmax\{Q_A(x_0), Q_A(y_0)\}$. Let $Q_A(x_0) = [\psi_1, \psi_2]$, $Q_A(y_0) = [\psi_3, \psi_4]$ and $Q_A(x_0 * y_0) = [t_1, t_2]$. Then $[t_1, t_2] > rmax\{[\psi_1, \psi_2], [\psi_3, \psi_4]\} = [\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}]$. So $t_1 > \max\{\psi_1, \psi_3\}$ and $t_2 > max\{\psi_2, \psi_4\}$. Let us consider,

$$\begin{aligned}
[\beta_1, \beta_2] &= \frac{1}{2}\Big[Q_A(x_0 * y_0) + rmax\{Q_A(x_0), Q_A(y_0)\}\Big] \\
&= \frac{1}{2}\Big[[t_1, t_2] + [\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}]\Big] \\
&= \Big[\frac{1}{2}(t_1 + \max\{\psi_1, \psi_3\}), \frac{1}{2}(t_2 + \max\{\psi_2, \psi_4\})\Big].
\end{aligned}$$

Therefore, $\max\{\psi_1, \psi_3\} < \beta_1 = \frac{1}{2}(t_1 + \max\{\psi_1, \psi_3\}) < t_1$ and $\max\{\psi_2, \psi_4\} < \beta_2 = \frac{1}{2}(t_2 + \max\{\psi_2, \psi_4\}) < t_2$. Hence, $[\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}] < [\beta_1, \beta_2] < [t_1, t_2]$ so that $x_0 * y_0 \notin L(Q_A : [t_1, t_2])$ which is a contradiction, since $Q_A(x_0) = [\psi_1, \psi_2] \leq [\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}] < [\beta_1, \beta_2]$ and $Q_A(y_0) = [\psi_3, \psi_4] \leq [\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}] < [\beta_1, \beta_2]$. Hence, $x_0 * y_0 \in L(Q_A : [t_1, t_2])$. Thus $Q_A(x * y) \leq rmax\{Q_A(x), Q_A(y)\}$ for all $x, y \in X$. □

Theorem 3.18. *Any subalgebra of X can be realized as both the upper $[s_1, s_2]$-level and lower $[t_1, t_2]$-level of some IVIF UP-subalgebra of X.*

Proof. Let P be an IVIF UP-subalgebra of X, and A be an IVIFS on X defined by

$$R_A(x) = \begin{cases} [\xi_1, \xi_2], & \text{if } x \in P \\ [0, 0], & \text{otherwise} \end{cases} \text{ and } \quad \mathrm{Q_A(x)} = \begin{cases} [\omega_1, \omega_2], & \text{if } x \in P \\ [1, 1], & \text{otherwise,} \end{cases}$$

for all $[\xi_1, \xi_2], [\omega_1, \omega_2] \in D[0,1]$ and $\xi_2 + \omega_2 \leq 1$. We consider the following cases:

Case (i) If $x, y \in P$, then $R_A(x) = [\xi_1, \xi_2]$, $Q_A(x) = [\omega_1, \omega_2]$ and $R_A(y) = [\xi_1, \xi_2]$, $Q_A(y) = [\omega_1, \omega_2]$. Thus, $R_A(x * y) = [\xi_1, \xi_2] = rmin\{[\xi_1, \xi_2], [\xi_1, \xi_2]\} = rmin\{R_A(x), R_A(y)\}$ and $Q_A(x * y) = [\omega_1, \omega_2] = rmax\{[\omega_1, \omega_2], [\omega_1, \omega_2]\} = rmax\{Q_A(x), Q_A(y)\}$.

Case (ii) If $x \in P$ and $y \notin P$ then $R_A(x) = [\xi_1, \xi_2]$, $Q_A(x) = [\omega_1, \omega_2]$ and $R_A(y) = [0, 0]$, $Q_A(y) = [1, 1]$. Thus, $R_A(x * y) \geq [0, 0] = rmin\{[\xi_1, \xi_2], [0, 0]\} = rmin\{R_A(x), R_A(y)\}$ and $Q_A(x*y) \leq [1, 1] = rmax\{[\omega_1, \omega_2], [1, 1]\} = rmax\{Q_A(x), Q_A(y)\}$.

Case (iii) If $x \notin P$ and $y \in P$ then $R_A(x) = [0, 0]$, $Q_A(x) = [1, 1]$, $R_A(y) = [\xi_1, \xi_2]$, $Q_A(y) = [\omega_1, \omega_2]$. Thus, $R_A(x * y) \geq [0, 0] = rmin\{[0, 0], [\xi_1, \xi_2]\} = rmin\{R_A(x), R_A(y)\}$ and $Q_A(x*y) \leq [1, 1] = rmax\{[1, 1], [\omega_1, \omega_2]\} = rmax\{Q_A(x), Q_A(y)\}$.

Case (iv) If $x \notin P$ and $y \notin P$ then $R_A(x) = [0, 0]$, $Q_A(x) = [1, 1]$ and $R_A(y) = [0, 0]$, $Q_A(y) = [1, 1]$. Now $R_A(x * y) \geq [0, 0] = rmin\{[0, 0], [0, 0]\} = rmin\{R_A(x), R_A(y)\}$ and $Q_A(x*y) \leq [1, 1] = rmax\{[1, 1], [1, 1]\} = rmax\{Q_A(x), Q_A(y)\}$.

Therefore, A is an IVIF UP-subalgebra of X. □

Theorem 3.19. *Let P be a subset of X and A be an IVIFS on X which is given in the proof of Theorem 3.18. If A be realized as lower level subalgebra and upper level subalgebra of some IVIF UP-subalgebra of X, then P is a IVIF UP-subalgebra of X.*

Proof. Let A be an IVIF UP-subalgebra of X, and $x, y \in P$. Then $R_A(x) = [\xi_1, \xi_2] = R_A(y)$ and $Q_A(x) = [\omega_1, \omega_2] = Q_A(y)$. Thus $R_A(x*y) \geq rmin\{R_A(x), R_A(y)\} = rmin\{[\xi_1, \xi_2], [\xi_1, \xi_2]\} = [\xi_1, \xi_2]$ and $Q_A(x*y) \leq rmax\{Q_A(x), Q_A(y)\} = rmax\{[\omega_1, \omega_2], [\omega_1, \omega_2]\} = [\omega_1, \omega_2]$, which imply that $x * y \in P$. Hence, the theorem. □

4. IVIF UP-ideals of UP-algebras

In this section we will define IVIF UP-ideal of UP-algebras and prove some propositions and theorems. In what follows, let X denote a UP-algebra unless otherwise specified.

Definition 4.1. *An IVIFS $A = (R_A, Q_A)$ in X is called an IVIF UP-ideal of X if it satisfies:*

(*UP3*) $\quad R_A(0) \geq R_A(x)$ *and* $Q_A(0) \leq Q_A(x)$

(*UP4*) $\quad R_A(x * z) \geq rmin\{R_A(x * (y * z)), R_A(y)\}$

(*UP5*) $\quad Q_A(x * z) \leq rmax\{Q_A(x * (y * z)), Q_A(y)\}$,

for all $x, y \in X$.

Example 4.2. Consider a UP-algebra $X=\{0,a,b,c,d\}$ with the following Cayley table

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Let $A=(R_A,Q_A)$ be an IVIFS in X defined as

$$R_A(x)=\begin{cases}[1,1], & \text{if } x\in\{0,a,b\}\\ [m_1,m_2], & \text{if } x\in\{c,d\}\end{cases} \quad \text{and} \quad \mathrm{Q_A(x)}=\begin{cases}[0,0], & \text{if } x\in\{0,a,b\}\\ [n_1,n_2], & \text{if } x\in\{c,d\},\end{cases}$$

where $[m_1,m_2]$, $[n_1,n_2]\in D[0,1]$ and $m_2+n_2\leq 1$. By routine calculations we get A is an IVIF UP-ideal of X.

Lemma 4.3. *Let $A=(R_A,Q_A)$ be an IVIF UP-ideal of X. If $x,y\in X$ is such that $y\leq x$, then $R_A(x)\geq R_A(y)$ and $Q_A(x)\leq Q_A(y)$.*

Proof. It is immediate and is omitted. □

Lemma 4.4. *Let $A=(R_A,Q_A)$ be an IVIF UP-ideal of X and $x,y,z,q\in X$. If $x\leq q*(y*z)$ then $R_A(x*z)\geq rmin\{R_A(q),R_A(y)\}$ and $Q_A(x*z)\leq rmax\{Q_A(q),Q_A(y)\}$.*

Proof. Let $x,y,z,q\in X$ such that $x\leq q*(y*z)$. Then $x*(q*(y*z))=0$ and thus $R_A(x*z)\geq rmin\{R_A(x*(y*z)),R_A(y)\}\geq rmin\{rmin\{R_A\{(x*(q*(y*z))),R_A(q)\},R_A(y)\}=rmin\{rmin\{R_A(0),R_A(q)\},R_A(y)\}=rmin\{R_A(q),R_A(y)\}$ and $Q_A(x*z)\leq rmax\{Q_A(x*(y*z)),Q_A(y)\}\leq rmax\{rmax\{Q_A\{(x*(q*(y*z))),Q_A(q)\},Q_A(y)\}=rmax\{rmax\{Q_A(0),Q_A(q)\},Q_A(y)\}=rmax\{Q_A(q),Q_A(y)\}$. □

Corollary 4.5. *Let $A=(R_A,Q_A)$ be an IVIF UP-ideal of X and $x,y,z\in X$. If $x\leq y*z$ then $R_A(x*z)\geq R_A(y)$ and $Q_A(x*z)\leq Q_A(y)$.*

Proof. Let $x,y,z\in X$ be such that $x\leq y*z$. Then by putting $q=0$ in Lemma 4.4 we have $x*(0*(y*z))=0$ and thus $R_A(x*z)\geq rmin\{R_A(0),R_A(y)\}=R_A(y)$ and $Q_A(x*z)\leq rmax\{Q_A(0),Q_A(y)\}=Q_A(y)$. □

Theorem 4.6. *Every IVIF UP-ideal of a UP-algebra X is an IVIF UP-subalgebra of X.*

Proof. Let $A=(R_A,Q_A)$ is an IVIF UP-ideal of X and $x,y\in X$. By Proposition 2.2, we have $x\leq y*x$. It follows from Lemma 4.3 that $R_A(y*x)\geq R_A(x)\geq rmin\{R_A(y),R_A(x)\}$ and $Q_A(y*x)\leq Q_A(x)\leq rmax\{Q_A(y),Q_A(x)\}$. Hence $A=(R_A,Q_A)$ is an IVIF UP-ideal of X. □

The converse of Theorem 4.6 may not be true. For example, the IVIF UP-subalgebra $A=(R_A,Q_A)$ in Example 3.2 is not an IVIF UP-ideal of X since $R_A(b*c)=[0.1,0.2]<[0.5,0.6]=rmin\{R_A(b*(a*c)),R_A(a)\}$.

Theorem 4.7. *An IVIFSs $A = \{[R_{AL}, R_{AU}], [Q_{AL}, Q_{AU}]\}$ in X is an IVIF UP-ideal of X if and only if R_{AL}, R_{AU}, Q_{AL} and Q_{AU} are fuzzy UP-ideals of X.*

Proof. Since $R_{AL}(0) \geq R_{AL}(x)$, $R_{AU}(0) \geq R_{AU}(x)$, $Q_{AL}(0) \leq Q_{AL}(x)$ and $Q_{AU}(0) \leq Q_{AU}(x)$, therefore $R_A(0) \geq R_A(x)$ and $Q_A(0) \leq Q_A(x)$.

Let R_{AL} and R_{AU} are fuzzy UP-ideals of X. Let $x, y, z \in X$. Then

$$\begin{aligned} R_A(x * z) &= [R_{AL}(x * z), R_{AU}(x * z)] \\ &\geq [min\{R_{AL}(x * (y * z)), R_{AL}(y)\}, min\{R_{AU}(x * (y * z)), R_{AU}(y)\}] \\ &= rmin\Big\{[R_{AL}(x * (y * z)), R_{AU}(x * (y * z))], [R_{AL}(y), R_{AU}(y)]\Big\} \\ &= rmin\{R_A(x * (y * z)), R_A(y)\}. \end{aligned}$$

Let Q_{AL} and Q_{AU} are fuzzy UP-ideals of X and $x, y \in X$. Then

$$\begin{aligned} Q_A(x * z) &= [Q_{AL}(x * z), Q_{AU}(x * z)] \\ &\leq [\max\{Q_{AL}(x * (y * z)), Q_{AL}(y)\}, \max\{Q_{AU}(x * (y * z)), Q_{AU}(y)\}] \\ &= rmax\Big\{[Q_{AL}(x * (y * z)), Q_{AU}(x * (y * z))], [Q_{AL}(y), Q_{AU}(y)]\Big\} \\ &= rmax\{Q_A(x * (y * z)), Q_A(y)\}. \end{aligned}$$

Hence, $A = \Big\{[R_{AL}, R_{AU}], [Q_{AL}, Q_{AU}]\Big\}$ is an IVIF UP-ideal of X.

Conversely, assume that, A is an IVIF UP-ideal of X. For any $x, y \in X$, we have $[R_{AL}(x * z), R_{AU}(x * z)] = R_A(x * z) \geq rmin\{R_A(x * (y * z)), R_A(y)\} = rmin\{[R_{AL}(x * (y * z)), R_{AU}(x * (y * z))], [R_{AL}(y), R_{AU}(y)]\} = [\min\{R_{AL}(x * (y * z)), R_{AL}(y)\}, \min\{R_{AU}(x * (y * z)), R_{AU}(y)\}]$ and $[Q_{AL}(x * z), Q_{AU}(x * z)] = Q_A(x * z) \leq rmax\{Q_A(x * (y * z)), Q_A(y)\} = rmax\{[Q_{AL}(x * (y * z)), Q_{AU}(x * (y * z))], [Q_{AL}(y), Q_{AU}(y)]\} = [\max\{Q_{AL}(x * (y * z)), Q_{AL}(y)\}, \max\{Q_{AU}(x * (y * z)), Q_{AU}(y)\}]$. Thus, $R_{AL}(x * z) \geq \min\{R_{AL}(x * (y * z)), R_{AL}(y)\}, R_{AU}(x * z) \geq \min\{R_{AU}(x * (y * z)), R_{AU}(y)\}, Q_{AL}(x * z) \leq \max\{Q_{AL}(x * (y * z)), Q_{AL}(y)\}, Q_{AU}(x * z) \leq \max\{Q_{AU}(x * (y * z)), Q_{AU}(y)\}$. Hence, R_{AL}, R_{AU}, Q_{AL} and Q_{AU} are fuzzy UP-ideals of X. $\square$

Theorem 4.8. *Let A_1 and A_2 be two IVIF UP-ideals of a UP-algebras X. Then $A_1 \cap A_2$ is also an IVIF UP-ideal of UP-algebra X.*

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2. Now, $R_{A_1 \cap A_2}(0) = R_{A_1 \cap A_2}(x * x) \geq rmin\{R_{A_1 \cap A_2}(x), R_{A_1 \cap A_2}(x)\} = R_{A_1 \cap A_2}(x)$ and $Q_{A_1 \cap A_2}(0) = Q_{A_1 \cap A_2}(x * x) \leq rmin\{Q_{A_1 \cap A_2}(x), Q_{A_1 \cap A_2}(x)\} = Q_{A_1 \cap A_2}(x)$. Also,

$$\begin{aligned} R_{A_1 \cap A_2}(x * z) &= [R_{(A_1 \cap A_2)L}(x * z), R_{(A_1 \cap A_2)U}(x * z)] \\ &\geq [\min(R_{(A_1 \cap A_2)L}(x * (y * z)), R_{(A_1 \cap A_2)L}(y)), \end{aligned}$$

$$\begin{aligned}
& \min(R_{(A_1\cap A_2)U}(x*(y*z)), R_{(A_1\cap A_2)U}(y))] \\
&= rmin\{R_{A_1\cap A_2}(x*(y*z)), R_{A_1\cap A_2}(y)\} \\
\text{and } Q_{A_1\cup A_2}(x*z) &= [Q_{(A_1\cup A_2)L}(x*z), Q_{(A_1\cup A_2)U}(x*z)] \\
&\leq [\max(Q_{(A_1\cup A_2)L}(x*(y*z)), Q_{(A_1\cup A_2)L}(y)), \\
& \max(Q_{(A_1\cup A_2)U}(x*(y*z)), Q_{(A_1\cup A_2)U}(y))] \\
&= rmax\{Q_{A_1\cup A_2}(x*(y*z)), Q_{A_1\cup A_2}(y)\}.
\end{aligned}$$

Hence, $A_1 \cap A_2$ is also an IVIF UP-ideal of UP-algebra X. □

Corollary 4.9. *Intersection of any family of IVIF UP-ideals of X is again an IVIF UP-ideal of X.*

Corollary 4.10. *If A is an IVIF UP-ideal of X then $\overline{A}$ is also an IVIF UP-ideal of X.*

Theorem 4.11. *If $A = (R_A, Q_A)$ is an IVIF UP-ideal of a UP-algebra X, then*

(i) $\bigoplus A$, *and*

(ii) $\bigotimes A$, *both are IVIF UP-ideals of UP-algebra X.*

Proof. For (i), it is sufficient to show that $\overline{R}_A$ satisfies the second part of the conditions (UP3) and (UP5). We have $\overline{R}_A(0) = 1-R_A(0) \leq 1-R_A(x) \leq \overline{R}_A(x)$. Let $x, y \in X$. Then $\overline{R}_A(x*z) = 1-R_A(x*z) \leq 1-rmin\{R_A(x*(y*z)), R_A(y)\} = rmax\{1-R_A(x*(y*z)), 1-R_A(y)\} = rmax\{\overline{R}_A(x*(y*z)), \overline{R}_A(y)\}$. Hence, $\bigoplus A$ is an IVIF UP-ideal of UP-subalgebra X.

For (ii), it is sufficient to show that $\overline{Q}_A$ satisfies the first part of the conditions (UP3) and (UP4). We have $\overline{Q}_A(0) = 1 - Q_A(0) \geq 1 - \overline{Q}_A(x) \geq \overline{Q}_A(x)$. Let $x, y \in X$. Then $\overline{Q}_A(x*z) = 1-Q_A(x*z) \geq 1-rmax\{Q_A(x*(y*z)), Q_A(y)\} = rmin\{1-Q_A(x*(y*z)), 1-Q_A(y)\} = rmin\{\overline{Q}_A(x*(y*z)), \overline{Q}_A(y)\}$. Hence, $\bigotimes A$ is an IVIF UP-ideal of UP-algebra X. □

Theorem 4.12. *An IVIFS A is an IVIF UP-ideal of X if and only if the sets $U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ are either empty or UP-ideal of X for every $[s_1, s_2], [t_1, t_2] \in D[0, 1]$.*

Proof. Suppose that $A = (R_A, Q_A)$ is an IVIF UP-ideal of X. Let $U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ be non-empty subset of X. Let $[s_1, s_2] \in D[0, 1]$ and $x, y, z \in X$ be such that $x * (y * z) \in U(R_A : [s_1, s_2])$ and $y \in U(R_A : [s_1, s_2])$. Then $R_A(x * z) \geq rmin\{R_A(x * (y * z)), R_A(y)\} \geq [s_1, s_2]$. Thus $x * z \in U(R_A : [s_1, s_2])$. Hence, $U(R_A : [s_1, s_2])$ is a UP-ideal of X.

Let $[t_1, t_2] \in D[0, 1]$ and $x, y, z \in X$ be such that $x * (y * z) \in L(Q_A : [t_1, t_2])$ and $y \in L(Q_A : [t_1, t_2])$. Then $Q_A(x * z) \leq rmax\{Q_A(x * (y * z)), Q_A(y)\} \leq [t_1, t_2]$. Thus $x * z \in L(Q_A : [t_1, t_2])$. Hence, $L(Q_A : [t_1, t_2])$ is a UP-ideal of X.

Conversely, assume that each non-empty level subset $U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ are UP-ideals of X. If there exist $\alpha, \beta, \gamma \in X$ such that $R_A(\alpha *$

$\gamma) < rmin\{R_A(\alpha * (\beta * \gamma)), R_A(\beta)\}$, then by taking $[s_1', s_2'] = \frac{1}{2}\Big[R_A(\alpha * \gamma) + rmin\{R_A(\alpha * (\beta * \gamma), R_A(\beta)\}\Big]$, it follows that $\alpha * (\beta * \gamma) \in U(R_A : [s_1', s_2'])$ and $\beta \in U(R_A : [s_1', s_2'])$, but $\alpha * \gamma \notin U(R_A : [s_1', s_2'])$, which is a contradiction. Hence, $U(R_A : [s_1', s_2'])$ is not UP-ideal of X.

Again if there exist $\lambda, \delta, \tau \in X$ such that $Q_A(\lambda * \tau) > rmax\{Q_A(\lambda * (\delta * \tau)), Q_A(\delta)\}$, then by taking $[t_1', t_2'] = \frac{1}{2}\Big[Q_A(\lambda*\tau)+rmax\{Q_A(\lambda*(\delta*\tau)), Q_A(\delta)\}\Big]$, it follows that $\lambda * (\delta * \tau) \in U(Q_A : [t_1', t_2'])$ and $\delta \in L(Q_A : [t_1', t_2'])$, but $\lambda * \tau \notin L(Q_A : [t_1', t_2'])$, which is a contradiction. Hence, $L(Q_A : [t_1', t_2'])$ is not UP-ideal of X.

Hence, $A = (R_A, Q_A)$ is an IVIF UP-ideal of X since it satisfies (UP3) and (UP4). □

5. Images and preimages of IVIF UP-subalgebras and UP-ideals

In this section we will present some results on images and preimages of IVIF UP-subalgebras and UP-ideals in UP-algebras.

Let f be a mapping from a set X into a set Y. Let $B = (R_B, Q_B)$ be an IVIFS in Y. Then the inverse image of B, is defined as $f^{-1}(B) = (f^{-1}(R_B), f^{-1}(Q_B))$ with the membership function and non-membership function respectively are given by $f^{-1}(R_B)(x) = R_B(f(x))$ and $f^{-1}(Q_B)(x) = Q_B(f(x))$. It can be shown that $f^{-1}(B)$ is an IVIFS.

Theorem 5.1. *Let $f : X \to Y$ be a homomorphism of UP-algebras. If $B = (R_B, Q_B)$ is an IVIF UP-subalgebra of Y, then the preimage $f^{-1}(B)=(f^{-1}(R_B), f^{-1}(Q_B))$ of B under f is an IVIF UP-subalgebra of X.*

Proof. Assume that B is an IVIF UP-subalgebra of Y and $x, y \in X$. Then $f^{-1}(R_B)(x*y) = R_B(f(x*y)) = R_B(f(x)*f(y)) \geq rmin\{R_B(f(x), R_B(f(y))\} = rmin\{f^{-1}(R_B)(x), f^{-1}(R_B)(y)\}$ and $f^{-1}(Q_B)(x*y) = Q_B(f(x*y)) = Q_B(f(x)* f(y)) \leq rmax\{Q_B(f(x), Q_B(f(y))\} = rmax\{f^{-1}(Q_B)(x), f^{-1}(Q_B)(y)\}$. Therefore, $f^{-1}(B)$ is an IVIF UP-subalgebra of X. □

Definition 5.2. *An IVIFS A in the UP-algebra X is said to have the rsup-property and rinf-property if for any subset T of X there exist $t_0 \in T$ such that $R_A(t_0) = rsup_{t_0 \in T} R_A(t)$ and $Q_A(t_0) = rinf_{t_0 \in T} Q_A(t)$ respectively.*

Definition 5.3. *Let f be a mapping from the set X to the set Y. If $A = (R_A, Q_A)$ is an IVIFS in X, then the image of A under f, denoted by $f(A)$, and is defined as*

$$f(A) = \{\langle x, f_{rsup}(R_A), f_{rinf}(Q_A)\rangle : x \in Y\},$$

where

$$f_{rsup}(R_A)(y) = \begin{cases} rsup_{x \in f^{-1}(y)} R_A(x), & \text{if} f^{-1}(y) \neq \phi \\ [0,0], & \text{otherwise} \end{cases}$$

and

$$f_{rinf}(Q_A)(y) = \begin{cases} rinf_{x\in f^{-1}(y)} Q_A(x), & \text{if} f^{-1}(y) \neq \phi \\ [1,1], & \text{otherwise.} \end{cases}$$

Theorem 5.4. *Let $f : X \to Y$ be a homomorphism from a UP-algebra X onto a UP-algebra Y. If $A = (R_A, Q_A)$ is an IVIF UP-subalgebra of X, then the image $f(A) = \{\langle x, f_{rsup}(R_A), f_{rinf}(Q_A)\rangle : x \in Y\}$ of A under f is an IVIF UP-subalgebra of Y.*

Proof. Let $A = (R_A, Q_A)$ be an IVIF UP-subalgebra of X and let $y_1, y_2 \in Y$. We know that, $\{x_1 * x_2 : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X : x \in f^{-1}(y_1 * y_2)\}$. Now,

$$\begin{aligned} f_{rsup}(R_A)(y_1 * y_2) &= rsup\{R_A(x) : x \in f^{-1}(y_1 * y_2)\} \\ &\geq rsup\{R_A(x_1 * x_2) : x_1 \in f^{-1}(y_1) \ \ and \ \ x_2 \in f^{-1}(y_2)\} \\ &\geq rsup\{rmin\{R_A(x_1), R_A(x_2)\} : x_1 \in f^{-1}(y_1) \ \ and \ \ x_2 \in f^{-1}(y_2)\} \\ &= rmin\{rsup\{R_A(x_1) : x_1 \in f^{-1}(y_1)\}, rsup\{R_A(x_2) : x_2 \in f^{-1}(y_2)\}\} \\ &= rmin\{f_{rsup}(R_A)(y_1), f_{rsup}(R_A)(y_2)\} \end{aligned}$$

and

$$\begin{aligned} f_{rinf}(Q_A)(y_1 * y_2) &= rinf\{Q_A(x) : x \in f^{-1}(y_1 * y_2)\} \\ &\leq rinf\{Q_A(x_1 * x_2) : x_1 \in f^{-1}(y_1) \ \ and \ \ x_2 \in f^{-1}(y_2)\} \\ &\leq rinf\{rmax\{Q_A(x_1), Q_A(x_2)\} : x_1 \in f^{-1}(y_1) \ \ and \ \ x_2 \in f^{-1}(y_2)\} \\ &= rmax\{rinf\{Q_A(x_1) : x_1 \in f^{-1}(y_1)\}, rinf\{Q_A(x_2) : x_2 \in f^{-1}(y_2)\}\} \\ &= rmax\{f_{rinf}(Q_A)(y_1), f_{rinf}(Q_A)(y_2)\}. \end{aligned}$$

Hence, $f(A) = \{\langle x, f_{rsup}(R_A), f_{rinf}(Q_A)\rangle : x \in Y\}$ is an IVIF UP-subalgebra of Y. $\square$

Theorem 5.5. *Let $f : X \to Y$ be a homomorphism of UP-algebras. If $B = (R_B, Q_B)$ is an IVIF UP-ideal of Y, then the pre-image $f^{-1}(B) = (f^{-1}(R_B), f^{-1}(Q_B))$ of B under f in X is an IVIF UP-ideal of X.*

Proof. For all $x \in X$ $f^{-1}(R_B)(x) = R_B(f(x)) \leq R_B(0) = R_B(f(0)) = f^{-1}(R_B)(0)$ and $f^{-1}(Q_B)(x) = Q_B(f(x)) \geq Q_B(0) = Q_B(f(0)) = f^{-1}(Q_B)(0)$. Let $x, y \in X$. Then $f^{-1}(R_B)(x * z) = R_B(f(x * z)) = R_B(f(x) * f(z)) \geq rmin\{R_B(f(x)*(f(y)*f(z))), R_B(f(y))\} = rmin\{R_B(f\ (x*(y*z))), R_B(f(y))\} = rmin\{f^{-1}(R_B)(x * (y * z)), f^{-1}(R_B)(y)\}$ and $f^{-1}(Q_B)(x * z) = Q_B(f(x * z)) = Q_B(f(x)*f(z)) \leq rmax\{Q_B(f(x)*(f(y)*f(z))), Q_B(f(y))\} = rmax\{Q_B(f(x*(y*z))), Q_B(f(y))\} = rmax\{f^{-1}(Q_B)(x*(y*z)), f^{-1}(Q_B)(y)\}$. Hence, $f^{-1}(B)$ is an IVIF UP-ideal of X. $\square$

Theorem 5.6. *Let $f : X \to Y$ be an epimorphism of UP-algebras. Then B is an IVIF UP-ideal of Y, if $f^{-1}(B) = (f^{-1}(R_B), f^{-1}(Q_B))$ of B under f in X is an IVIF UP-ideal of X.*

Proof. For any $x \in Y$, $\exists\, a \in X$ such that $f(a) = x$. Then $R_B(x) = R_B(f(a)) = f^{-1}(R_B)(a) \leq f^{-1}(R_B)(0) = R_B(f(0)) = R_B(0)$ and $Q_B(x) = Q_B(f(a)) = f^{-1}(Q_B)(a) \geq f^{-1}(Q_B)(0) = Q_B(f(0)) = Q_B(0)$. Let $x, y, z \in Y$. Then $f(a) = x$, $f(b) = y$ and $f(c) = z$ for some $a, b, c \in X$. Thus $R_B(x * z) = R_B(f(a) * f(c)) = M_B(f(a * c)) = f^{-1}(R_B)(a * c) \geq rmin\{f^{-1}(R_B)(a * (b * c)), f^{-1}(R_B)(b)\} = rmin\{R_B(f(a * (b * c))), R_B(f(b))\} = rmin\{R_B(f(a) * (f(b) * f(c))), R_B(f(b))\} = rmin\{R_B(x * (y * z)), R_B(y)\}$ and $Q_B(x * z) = Q_B(f(a) * f(c)) = N_B(f(a * c)) = f^{-1}(Q_B)(a * c) \leq rmax\{f^{-1}(Q_B)(a * (b * c)), f^{-1}(Q_B)(b)\} = rmax\{Q_B(f(a * (b * c))), Q_B(f(b))\} = rmax\{Q_B\ (f(a) * (f(b) * f(c))), Q_B(f(b))\} = rmax\{Q_B(x * (y * z)), Q_B(y)\}$. Then B is an IVIF UP-ideal of Y. □

6. Equivalence relations on IVIF UP-ideals

Let IVIFI(X) denote the family of all interval-valued intuitionistic fuzzy ideals of X and let $\rho = [\rho_1, \rho_2] \in D[0, 1]$. Define binary relations U^ρ and L^ρ on IVIFI(X) as follows:

$$(A, B) \in U^\rho \Leftrightarrow U(R_A : \rho) = U(R_B : \rho) \text{ and}$$
$$(A, B) \in L^\rho \Leftrightarrow L(Q_A : \rho) = L(Q_B : \rho)$$

respectively, for A=(R_A, Q_A) and B=(R_B, Q_B) in IVIFI(X). Then clearly U^ρ and L^ρ are equivalence relations on IVIFI(X). For any A=$(R_A, Q_A)$$\in$$IVIFI(X)$, let $[A]_{U^\rho}$ (respectively, $[A]_{L^\rho}$) denote the equivalence class of A modulo U^ρ (respectively, L^ρ), and denote by IVIFI(X)/U^ρ (respectively, IVIFI(X)/L^ρ) the collection of all equivalence classes modulo U^ρ (respectively, L^ρ), i.e.,

$$IVIFI(X)/U^\rho := \{[A]_{U^\rho} | A = (R_A, Q_A) \in IVIFI(X)\},$$

respectively,

$$IVIFI(X)/L^\rho := \{[A]_{L^\rho} | A = (R_A, Q_A) \in IVIFI(X)\}.$$

These two sets are also called the quotient sets.

Now let $T(X)$ denote the family of all ideals of X and let $\rho = [\rho_1, \rho_2] \in D[0, 1]$. Define mappings f_ρ and g_ρ from IVIFI(X) to $T(X) \cup \{\phi\}$ by $f_\rho(A) = U(R_A : \rho)$ and $g_\rho(A) = L(Q_A : \rho)$, respectively, for all $A = (R_A, Q_A) \in IVIFI(X)$. Then f_ρ and g_ρ are clearly well-defined.

Theorem 6.1. *For any $\rho = [\rho_1, \rho_2] \in D[0, 1]$, the maps f_ρ and g_ρ are surjective from IVIFI(X) to $T(X) \cup \{\phi\}$.*

Proof. Let $\rho = [\rho_1, \rho_2] \in D[0, 1]$. Note that $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1})$ is in IVIFI(X), where $\mathbf{0}$ and $\mathbf{1}$ are interval-valued fuzzy sets in X defined by $\mathbf{0}(x) = [0, 0]$ and $\mathbf{1}(x) = [1, 1]$ for all $x \in X$. Obviously $f_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho)$ $=U([0, 0] : [\rho_1, \rho_2])$$= \phi$ $=L([1, 1] : [\rho_1, \rho_2])$ $=L(\mathbf{1} : \rho)$$=g_\rho(\mathbf{0}_\sim)$. Let $P(\neq \phi) \in IVIFI(X)$. For $P_\sim = (\chi_P, \overline{\chi}_P) \in IVIFI(X)$, we have $f_\rho(P_\sim)$$=U(\chi_P : \rho) = P$ and $g_\rho(P_\sim) = L(\overline{\chi}_P : \rho) = P$. Hence f_ρ and g_ρ are surjective. □

Theorem 6.2. *The quotient sets IVIFI(X)/U^ρ and IVIFI(X)/L^ρ are equipotent to $T(X) \cup \{\phi\}$ for every $\rho \in D[0,1]$.*

Proof. For $\rho \in D[0,1]$ let f_ρ^* (respectively, g_ρ^*) be a map from IVIFI(X)/U^ρ (respectively, IVIFI(X)/L^ρ) to $T(X)\cup\{\phi\}$ defined by $f_\rho^*([A]_{U^\rho}) = f_\rho(A)$ (respectively, $g_\rho^*([A]_{U^\rho}) = g_\rho(A)$) for all $A = (R_A, Q_A) \in IVIFI(X)\}$. If $U(R_A : \rho) = U(R_B : \rho)$ and $L(Q_A : \rho) = L(Q_B : \rho)$ for $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ in IVIFI(X), then $(A,B) \in U^\rho$ and $(A,B) \in L^\rho$; hence $[A]_{U^\rho} = [B]_{U^\rho}$ and $[A]_{L^\rho} = [B]_{L^\rho}$. Therefore the maps f_ρ^* and g_ρ^* are injective. Now let $P(\neq \phi) \in IVIFI(X)$. For $P_\sim = (\chi_P, \overline{\chi}_P) \in IVIFI(X)$, we have

$$f_\rho^*([P_\sim]_{U^\rho}) = f_\rho(P_\sim) = U(\chi_P : \rho) = P,$$

and

$$g_\rho^*([P_\sim]_{L^\rho}) = g_\rho(P_\sim) = L(\overline{\chi}_P : \rho) = P.$$

Finally, for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IVIFI(X)$ we get

$$f_\rho^*([\mathbf{0}_\sim]_{U^\rho}) = f_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) = \phi$$

and

$$g_\rho^*([\mathbf{0}_\sim]_{L^\rho}) = g_\rho(\mathbf{0}_\sim) = L(\mathbf{1} : \rho) = \phi.$$

This shows that f_ρ^* and g_ρ^* are surjective. This completes the proof. □

For any $\rho \in D[0,1]$, we define another relation R^ρ on IVIFI(X) as follows:

$$(A,B) \in R^\rho \Leftrightarrow U(R_A : \rho) \cap L(Q_A : \rho) = U(R_B : \rho) \cap L(Q_B : \rho),$$

for any $A = (R_A, Q_A)$ and $B = (R_B, Q_B) \in IVIFI(X)$. Then the relation R^ρ is an equivalence relation on IVIFI(X).

Theorem 6.3. *For any $\rho \in D[0,1]$, the maps $\psi_\rho : IVIFI(X) \to T(X) \cap \{\phi\}$ defined by $\psi_\rho(A) = f_\rho(A) \cap g_\rho(A)$ for each $A = (R_A, Q_A) \in X$ is surjective.*

Proof. Let $\rho \in D[0,1]$. For $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IVIFI(X)$,

$$\psi_\rho(\mathbf{0}_\sim) = f_\rho(\mathbf{0}_\sim) \cap g_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) \cap L(\mathbf{1} : \rho) = \phi.$$

For any $H \in IVIFI(X)$, there exists $H_\sim = (\chi_H, \overline{\chi}_H) \in IVIFI(X)$ such that

$$\psi_\rho(H_\sim) = f_\rho(H_\sim) \cap g_\rho(H_\sim) = U(\chi_H : \rho) \cap L(\overline{\chi}_H : \rho) = H.$$

This completes the proof. □

Theorem 6.4. *The quotient sets IVIFI(X)/R^ρ are equipotent to $T(X) \cup \{\phi\}$ for every $\rho \in D[0,1]$.*

Proof. For $\rho \in D[0,1]$, define a map $\psi_\rho^* : IVIFI(X)/R^\rho \to T(X) \cup \{\phi\}$ by $\psi_\rho^*([A]_{R^\rho}) = \psi_\rho(A)$ for all $[A]_{R^\rho} \in IVIFI(X)/R^\rho$. Assume that $\psi_\rho^*([A]_{R^\rho}) = \psi_\rho^*([B]_{R^\rho})$ for any $[A]_{R^\rho}$ and $[B]_{R^\rho} \in IVIFI(X)/R^\rho$. Then $f_\rho(A) \cap g_\rho(A) = f_\rho(B) \cap g_\rho(B)$, i.e., $U(R_A : \rho) \cap L(Q_A : \rho) = U(R_B : \rho) \cap L(Q_B : \rho)$. Hence

$(A,B) \in R^\rho$, and so $[A]_{R^\rho} = [B]_{R^\rho}$. Therefore the maps ψ_ρ^* are injective. Now for $\mathbf{0}_\sim = (\mathbf{0},\mathbf{1}) \in IVIFI(X)$ we have

$$\psi_\rho^*([\mathbf{0}_\sim]_{R^\rho}) = \psi_\rho(\mathbf{0}_\sim) = f_\rho(\mathbf{0}_\sim) \cap g_\rho(\mathbf{0}_\sim) = U(\mathbf{0}:\rho) \cap L(\mathbf{1}:\rho) = \phi.$$

If $H \in IVIFI(X)$, then for $H_\sim = (\chi_H, \overline{\chi}_H) \in IVIFI(X)$, we obtain

$$\psi_\rho^*([H_\sim]_{R^\rho}) = \psi_\rho(H_\sim) = f_\rho(H_\sim) \cap g_\rho(H_\sim) = U(\chi_H:\rho) \cap L(\overline{\chi}_H:\rho) = H.$$

Thus ψ_ρ^* is surjective. This completes the proof. □

7. Product of IVIF UP-subalgebras and UP-ideals

In this section we will provide some new definitions on cartesian product of IVIF UP-subalgebras and UP-ideals in UP-algebras.

Definition 7.1. *Let $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ be two IVIFSs of X and Y respectively. The cartesian product $A \times B = (R_A \times R_B, Q_A \times Q_B)$ of $X \times Y$ is defined by $(R_A \times R_B)(x,y) = rmin\{R_A(x), R_B(y)\}$ and $(Q_A \times Q_B)(x,y) = rmax\{Q_A(x), Q_B(y)\}$, where $R_A \times R_B : X \times Y \to D[0,1]$ and $Q_A \times Q_B : X \times Y \to D[0,1]$ for all $(x,y) \in X \times Y$.*

Remark 7.2. *Let X and Y be UP-algebras. We define $*$ on $X \times Y$ by $(x,y)*(u,v) = (x*u, y*v)$ for every (x,y), (u,v) belong to $X \times Y$, then clearly $(X \times Y, *, (0,0))$ is a UP-algebra.*

Definition 7.3. *An IVIFS $A \times B = (R_A \times R_B, Q_A \times Q_B)$ of $X \times Y$ is called an IVIF UP-subalgebra if it satisfies for all (x_1,y_1) and $(x_2,y_2) \in X \times Y$*

(i) $(R_A \times R_B)((x_1,y_1)*(x_2,y_2)) \geq rmin\{(R_A \times R_B)(x_1,y_1), (R_A \times R_B)(x_2,y_2)\}$,

(ii) $(Q_A \times Q_B)((x_1,y_1)*(x_2,y_2)) \leq rmax\{(Q_A \times Q_B)(x_1,y_1), (Q_A \times Q_B)(x_2,y_2)\}$.

Definition 7.4. *An IVIFS $A \times B = (R_A \times R_B, Q_A \times Q_B)$ of $X \times Y$ is called an IVIF UP-ideal if it satisfies for all (x_1,y_1), (x_2,y_2) and $(x_3,y_3) \in X \times Y$*

(i) $(R_A \times R_B)(0,0) \geq (R_A \times R_B)(x,y)$ *and* $(Q_A \times Q_B)(0,0) \leq (Q_A \times Q_B)(x,y)$,

(ii) $(R_A \times R_B)((x_1,y_1)*(x_3,y_3)) \geq rmin\{(R_A \times R_B)((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), (R_A \times R_B)(x_2,y_2)\}$ *and*

(iii) $(Q_A \times Q_B)((x_1,y_1)*(x_3,y_3)) \leq rmax\{(Q_A \times Q_B)((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), (Q_A \times Q_B)(x_2,y_2)\}$.

Theorem 7.5. *Let $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ be IVIF UP-subalgebras of X and Y respectively, then $A \times B$ is an IVIF UP-subalgebra of $X \times Y$.*

Proof. For any (x_1, y_1) and $(x_2, y_2) \in X \times Y$, we have

$$\begin{aligned}(R_A \times R_B)((x_1, y_1) * (x_2, y_2)) &= (R_A \times R_B)(x_1 * x_2, y_1 * y_2)\\ &= rmin\{R_A(x_1 * x_2), R_B(y_1 * y_2)\}\\ &\geq rmin\{rmin\{R_A(x_1), R_A(x_2)\}, rmin\{R_B(y_1), R_B(y_2)\}\}\\ &= rmin\{rmin\{R_A(x_1), R_B(y_1)\}, rmin\{R_A(x_2), R_B(y_2)\}\}\\ &= rmin\{(R_A \times R_B)(x_1, y_1), (R_A \times R_B)(x_2, y_2)\}\end{aligned}$$

and

$$\begin{aligned}(Q_A \times Q_B)((x_1, y_1) * (x_2, y_2)) &= (Q_A \times Q_B)(x_1 * x_2, y_1 * y_2)\\ &= rmax\{Q_A(x_1 * x_2), Q_B(y_1 * y_2)\}\\ &\leq rmax\{rmax\{Q_A(x_1), Q_A(x_2)\}, rmax\{Q_B(y_1), Q_B(y_2)\}\}\\ &= rmax\{rmax\{Q_A(x_1), Q_B(y_1)\}, rmax\{Q_A(x_2), Q_B(y_2)\}\}\\ &= rmax\{(Q_A \times Q_B)(x_1, y_1), (Q_A \times Q_B)(x_2, y_2)\}.\end{aligned}$$

Hence, $A \times B$ is an IVIF UP-subalgebra of $X \times Y$. □

Definition 7.6. *Let $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ be IVIF UP-subalgebras of X and Y respectively. For $[s_1, s_2], [t_1, t_2] \in D[0, 1]$, the set $U(R_A \times R_B : [s_1, s_2]) = \{(x, y) \in X \times Y | (R_A \times R_B)(x, y) \geq [s_1, s_2]\}$ is called upper $[s_1, s_2]$-level of $A \times B$ and $L(Q_A \times Q_B : [t_1, t_2]) = \{(x, y) \in X \times Y | (Q_A \times Q_B)(x, y) \leq [t_1, t_2]\}$ is called lower $[t_1, t_2]$-level of $A \times B$.*

Theorem 7.7. *For any IVIFS A and B, if $A \times B$ is an IVIF UP-subalgebra of $X \times Y$ then non-empty upper $[s_1, s_2]$-level cut $U(R_A \times R_B : [s_1, s_2])$ and non-empty lower $[t_1, t_2]$-level cut $L(Q_A \times Q_B : [t_1, t_2])$ are UP-subalgebras of $X \times Y$, for all $[s_1, s_2]$ and $[t_1, t_2] \in D[0, 1]$.*

Proof. Let A and B be such that $A \times B$ is an IVIF UP-subalgebra of $X \times Y$, therefore, $(R_A \times R_B)((x_1, y_1) * (x_2, y_2)) \geq rmin\{(R_A \times R_B)(x_1, y_1), (R_A \times R_B)(x_2, y_2)\}$ and $(Q_A \times Q_B)((x_1, y_1) * (x_2, y_2)) \leq rmax\{(Q_A \times Q_B)(x_1, y_1), (Q_A \times Q_B)(x_2, y_2)\}$, for all (x_1, y_1) and $(x_2, y_2) \in X \times Y$.

Again, let $(x_1, y_1), (x_2, y_2) \in X \times Y$ be such that (x_1, y_1) and $(x_2, y_2) \in U(R_A \times R_B : [s_1, s_2])$. Then, $(R_A \times R_B)((x_1, y_1) * (x_2, y_2)) \geq rmin\{(R_A \times R_B)(x_1, y_1), (R_A \times R_B)(x_2, y_2)\} \geq rmin([s_1, s_2], [s_1, s_2]) = [s_1, s_2]$. This implies, $((x_1, y_1) * (x_2, y_2)) \in U(R_A \times R_B : [s_1, s_2])$. Thus $U(R_A \times R_B : [s_1, s_2])$ is a UP-subalgebra of $X \times Y$. Similarly, $L(Q_A \times Q_B : [t_1, t_2])$ is a UP-subalgebra of $X \times Y$. □

Proposition 7.8. *Let A and B be IVIF UP-ideals of X, then $A \times B$ is an IVIF UP-ideal of $X \times X$.*

Proof. For any $(x, y) \in X \times X$, we have $(R_A \times R_B)(0, 0) = rmin\{R_A(0), R_B(0)\} \geq rmin\{R_A\ (x), R_B(y)\} = (R_A \times R_B)(x, y)$ and $(Q_A \times Q_B)(0, 0) = rmax\{Q_A(0), Q_B(0)\} \leq rmin\{Q_A(x),\ Q_B(y)\} = (Q_A \times Q_B)(x, y)$.

Let (x_1, y_1), (x_2, y_2), $(x_3, y_3) \in X \times X$. Then,

$$\begin{aligned}
&(R_A \times R_B)((x_1, y_1) * (x_3, y_3)) \\
&= (R_A \times R_B)(x_1 * x_3, y_1 * y_3) = rmin\{R_A(x_1 * x_3), R_B(y_1 * y_3)\} \\
&\geq rmin\{rmin\{R_A(x_1 * (x_2 * x_3)), R_A(x_2)\}, rmin\{R_B(y_1 * (y_2 * y_3)), R_B(y_2)\}\} \\
&= rmin\{rmin\{R_A(x_1 * (x_2 * x_3)), R_B(y_1 * (y_2 * y_3))\}, rmin\{R_A(x_2), R_B(y_2)\}\} \\
&= rmin\{(R_A \times R_B)(x_1 * (x_2 * x_3), y_1 * (y_2 * y_3)), (R_A \times R_B)(x_2, y_2)\} \\
&= rmin\{(R_A \times R_B)((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), (R_A \times R_B)(x_2, y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
&(Q_A \times Q_B)((x_1, y_1) * (x_3, y_3)) \\
&= (Q_A \times Q_B)(x_1 * x_3, y_1 * y_3) = rmax\{Q_A(x_1 * x_3), Q_B(y_1 * y_3)\} \\
&\leq rmax\{rmax\{Q_A(x_1 * (x_2 * x_3)), Q_A(x_2)\}, rmax\{Q_B(y_1 * (y_2 * y_3)), Q_B(y_2)\}\} \\
&= rmax\{rmax\{Q_A(x_1 * (x_2 * x_3)), Q_B(y_1 * (y_2 * y_3))\}, rmax\{Q_A(x_2), Q_B(y_2)\}\} \\
&= rmax\{(Q_A \times Q_B)(x_1 * (x_2 * x_3), y_1 * (y_2 * y_3)), (Q_A \times Q_B)(x_2, y_2)\} \\
&= rmax\{(Q_A \times Q_B)((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), (Q_A \times Q_B)(x_2, y_2)\}.
\end{aligned}$$

Hence, $A \times B$ is an IVIF UP-ideal of $X \times X$. □

Lemma 7.9. *If $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ are IVIF UP-ideals of X, then $\bigoplus(A \times B) = (R_A \times R_B, \overline{R}_A \times \overline{R}_B)$ is an IVIF UP-ideals of $X \times X$.*

Proof. Let $(R_A \times R_B)(x, y) = rmin\{R_A(x), R_B(y)\}$. Then $1-(\overline{R}_A \times \overline{R}_B)(x, y) = rmin\{1 - \overline{R}_A(x), 1 - \overline{R}_B(y)\}$. This implies, $1 - rmin\{1 - \overline{R}_A(x), 1 - \overline{R}_B(y)\} = (\overline{R}_A \times \overline{R}_B)(x, y)$. Therefore, $(\overline{R}_A \times \overline{R}_B)(x, y) = rmax\{\overline{R}_A(x), \overline{R}_B(y)\}$. Hence, $\bigoplus(A \times B) = (R_A \times R_B, \overline{R}_A \times \overline{R}_B)$ is an IVIF UP-ideal of $X \times X$. □

Lemma 7.10. *If $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ are IVIF UP-ideals of X, then $\bigotimes(A \times B) = (\overline{Q}_A \times \overline{Q}_B, Q_A \times Q_B)$ is an IVIF UP-ideal of $X \times X$.*

Proof. Let $(Q_A \times Q_B)(x, y) = rmax\{Q_A(x), Q_B(y)\}$. This implies, $1 - (\overline{Q}_A \times \overline{Q}_B)(x, y) = rmax\{1 - \overline{Q}_A(x), 1 - \overline{Q}_B(y)\}$. This is, $1 - rmax\{1 - \overline{Q}_A(x), 1 - \overline{Q}_B(y)\} = (\overline{Q}_A \times \overline{Q}_B)(x, y)$. Therefore, $(\overline{Q}_A \times \overline{Q}_B)(x, y) = rmin\{\overline{Q}_A(x), \overline{Q}_B(y)\}$. Hence, $\bigotimes(A \times B) = (\overline{Q}_A \times \overline{Q}_B, Q_A \times Q_B)$ is an IVIF UP-ideal of $X \times X$. □

By the above two lemmas, it is not difficult to verify that the following theorem is valid.

Theorem 7.11. *The IVIFSs $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ are IVIF UP-ideals of X if and only if $\bigoplus(A \times B) = (R_A \times R_B, \overline{R}_A \times \overline{R}_B)$ and $\bigotimes(A \times B) = (\overline{Q}_A \times \overline{Q}_B, Q_A \times Q_B)$ are IVIF UP-ideal of $X \times X$.*

Theorem 7.12. *For any IVIFS A and B, if $A \times B$ is an IVIF UP-ideals of $X \times X$ then the non-empty upper $[s_1, s_2]$-level cut $U(R_A \times R_B : [s_1, s_2])$ and the non-empty lower $[t_1, t_2]$-level cut $L(Q_A \times Q_B : [t_1, t_2])$ are UP-ideals of $X \times X$ for any $[s_1, s_2]$ and $[t_1, t_2] \in D[0, 1]$.*

Proof. Assume that A and B are IVIF UP-ideals of X. Let (a,b), (c,d), $(e,f) \in X \times X$ be such that $(a,b) * ((e,f) * (c,d))$, $(e,f) \in U(R_A \times R_B : [s_1, s_2])$. Then $(R_A \times R_B)((a,b) * (c,d)) \geq rmin\{(R_A \times R_B)(a,b) * ((e,f) * (c,d)), (R_A \times R_B)(e,f)\} \geq rmin([s_1,s_2],[s_1,s_2]) = [s_1,s_2]$. This implies, $(a,b) * (c,d) \in U(R_A \times R_B : [s_1,s_2])$. Thus $U(R_A \times R_B : [s_1,s_2])$ is a UP-ideal of $X \times X$. Similarly, $L(Q_A \times Q_B : [t_1,t_2])$ is a UP-ideal of $X \times X$. □

References

[1] K.T. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, Studies in fuzziness and soft computing, Vol. 35, Physica-Verlag, Heidelberg/New York, 1999.

[2] M. Bhowmik, T. Senapati and M. Pal, *Intuitionistic L-fuzzy ideals in BG-algebras*, Afrika Matematika, 25(3) (2014), 577-590.

[3] A. Iampan, *A new branch of the logical algebra: UP-algebras*, Manuscript submitted for publication, April 2014.

[4] K. Iseki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica, 23 (1978), 1-26.

[5] C. Jana, T. Senapati, M. Bhowmik and M. Pal, *On intuitionistic fuzzy G-subalgebras of G-algebras*, *Fuzzy Inf. Eng.*, 7(2) (2015), 195-209.

[6] C. Jana, M. Pal, T. Senapati and M. Bhowmik, *Atanassov's intutionistic L-fuzzy G-subalgebras of G-algebras, J. Fuzzy Math.*, 23(2) (2015), 195-209.

[7] B. Kesorn, K. Maimun, W. Ratbandan and A. Iampan, *Intuitionistic fuzzy sets in UP-algebras*, Itaian Journal of Pure and Applied Mathematics, 34 (2015), 339-364.

[8] C.B. Kim and H.S. Kim, *On BG-algebras, Demonstratio Mathematica*, 41 (2008), 497-505.

[9] J. Neggers and H.S. Kim, *On B-algebras*, Math. Vensik, 54 (2002), 21-29.

[10] T. Senapati, *Bipolar fuzzy structure of BG-subalgebras*, J. Fuzzy Math., 23(1) (2015), 209-220.

[11] T. Senapati, M. Bhowmik and M. Pal, *Triangular norm based fuzzy BG-algebras*, Afrika Matematika, 27(1-2) (2016), 187199.

[12] T. Senapati, M. Bhowmik, M. Pal and B. Davvaz, *Fuzzy translations of fuzzy H-ideals in BCK/BCI-algebras*, J. Indones. Math. Soc., 21(1) (2015), 45-58.

[13] T. Senapati, M. Bhowmik, M. Pal and B. Davvaz, *Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy subalgebras and ideals in BCK/BCI-algebras*, Eurasian Mathematical Journal, 6(1) (2015), 96-114.

[14] T. Senapati, M. Bhowmik and M. Pal, *Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy H-ideals in BCK/BCI-algebras*, Notes Intuition. Fuzzy Sets, 19(1) (2013), 32-47.

[15] T. Senapati, C. Jana, M. Bhowmik and M. Pal, *L-fuzzy G-subalgebras of G-algebras*, Journal of the Egyptian Mathematical Society, 23 (2015), 219-223.

[16] T. Senapati, M. Bhowmik and M. Pal, *Fuzzy dot structure of BG-algebras*, Fuzzy Inf. Eng., 6(3) (2014), 315-329.

[17] T. Senapati, C.S. Kim, M. Bhowmik and M. Pal, *Cubic subalgebras and cubic closed ideals of B-algebras*, Fuzzy Inf. Eng., 7(2) (2015), 129-149.

[18] T. Senapati, *Translations of intuitionistic fuzzy B-algebras,* Fuzzy Inf. Eng., 7(4) (2015), 389-404.

[19] T. Senapati, M. Bhowmik and M. Pal, *Interval-valued intuitionistic fuzzy BG-subalgebras*, J. Fuzzy Math., 20(3) (2012), 707-720.

[20] T. Senapati, M. Bhowmik and M. Pal, *Fuzzy dot subalgebras and fuzzy dot ideals of B-algebras*, J. Uncert. Syst., 8(1) (2014), 22-30.

[21] T. Senapati, M. Bhowmik and M. Pal, *Intuitionistic fuzzifications of ideals in BG-algebras*, Mathematica Aeterna, 2(9) (2012), 761-778.

[22] T. Senapati and K.P. Shum, *Atanassov's intuitionistic fuzzy bi-normed KU-ideal of a KU-algebra*, J. Intell. Fuzzy Systems, 30 (2016), 1169-1180.

[23] T. Senapati and K.P. Shum, *Atanassovs intuitionistic fuzzy bi-normed KU-subalgebrs of a KU-algebra*, Missouri J. Math. Sci., 29(1) (2017), 92-112.

[24] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw and A. Iampan, *Fuzzy sets in UP-algebras*, Ann. Fuzzy Math. Inform., accepted.

[25] L.A. Zadeh, *Fuzzy sets*, Inform. and Control, 8(3) (1965), 338-353.

Accepted: 3.06.2017